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## Energy eigenvalues of quartic oscillators in $d \leq 3$ dimensions

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**Abstract.** We present a simple approximate analytical expression for the energy eigenvalues  $E^{(d)}$  of the pure quartic oscillator in  $d \leq 3$  dimensions. The formula reproduces with high accuracy the results from accurate numerical computations reported in the literature. The eigenvalue  $E_{n,l}^{(d)}$  for given  $n$  is seen to decrease as  $l$  increases contrary to the prediction of the Quigg–Rosner formula (for  $d = 3$ ) which is incorrect in this respect. We also give a generalisation (from the pure quartic to the quartic anharmonic oscillator) of our formula, good for any  $d$ . This formula involves four parameters (one less in the pure quartic case) of which all but one are obtainable by consideration of the wkb limit; the values of all the parameters are independent of the dimension  $d$ .

### 1. Introduction

The literature available on higher-dimensional anharmonic oscillators is rather limited, when compared with that available for the one-dimensional system. Exhaustive numerical tables as well as many analytical formulae (with limited domains of validity) have been reported in the literature for the energy spectra associated with anharmonic potentials  $V = \omega^2 x^2 + \lambda x^{2\alpha}$  in one dimension, both for the case  $\omega \neq 0$  and  $\omega = 0$ † (Bazley and Fox 1961, Chan and Stelman 1963, Biswas *et al* 1971, 1973, Lakshmanan and Prabhakaran 1973, Hioe and Montroll 1975, Hioe *et al* 1976, 1978, Banerjee *et al* 1978, Banerjee 1978, Ginsburg and Montroll 1978, Caswell 1979, Richardson and Blankenbecler 1979, Halliday and Suranyi 1980, Killingbeck 1981, Mathews *et al* 1981). Bell *et al* (1970a, b, see also Lu and Nigam 1969) have computed numerically the first 100 energy levels of the two- and three-dimensional quartic oscillators ( $V = \lambda (r \cdot r)^2$ ) by diagonalising large dimensional matrices of order 800. More recently Pasupathy and Singh (1980), Quigg and Rosner (1979) and Lakshmanan and Kaliappan (1980) have derived approximate formulae for the three-dimensional quartic or quartic anharmonic oscillator (a system of interest in charmonium physics) via wkb or phase integral techniques. While Pasupathy and Singh's wkb results are for s waves, and are therefore equivalent to results for the one-dimensional system, the generalisation of the s-wave result to  $l > 0$  by Quigg and Rosner is incorrect. (Their claim that for large but fixed  $n$ , the energy will depend only on  $n + l$  but not on  $n$  and  $l$  separately is at variance with the formula given in this paper as well as the numerical results known for the system.) Lakshmanan and Kaliappan have adopted the semiclassical Bohr–Sommerfeld quantisation scheme and as such their analytical formula is expected to be

† The literature on the one-dimensional oscillator is so vast that what we have given is only a selective sample over the years in which a variety of methods has been used to obtain results.



satisfactory only in the large- $n$  regime. Since no numerical results have been given by them, there is no information on the degree of accuracy of their formula. However for two- and three-dimensional oscillators a single analytical formula which is capable of yielding the energy values with a high degree of accuracy for all values of the quantum numbers has so far been lacking. It is our aim in this paper to present a simple and accurate formula for the energy levels  $E_{n,l}^{(d)}$  of the pure quartic oscillator ( $V = (\mathbf{r} \cdot \mathbf{r})^2$ ) in  $d \leq 3$  dimensions. It reproduces with considerable accuracy the numerically computed values reported by earlier workers, for all values of level number  $n$  and the angular momentum quantum number  $l$ . To our knowledge no such formula has so far been published in the literature. We wish to emphasise that we consider the formula as an end in itself. No great profundity is involved in its derivation, but that does not detract in any way from its utility as a means of estimating  $E_{n,l}^{(d)}$  quickly for any desired  $n$ ,  $l$  and  $d$ . This estimate could be very useful as a starting point in various types of numerical schemes intended for calculation of eigenvalues to arbitrary accuracy. It is in the same spirit that we also write down, towards the end of the paper, a formula for the energy levels of the quartic *anharmonic* oscillator ( $V = \frac{1}{2}m\omega^2 \mathbf{r} \cdot \mathbf{r} + \lambda (\mathbf{r} \cdot \mathbf{r})^2$ ) in  $d$  dimensions. This generalises the expression of the  $d = 1$  case presented in a recent note by Mathews *et al* (1981).

The approximate expression that we give below for the energy eigenvalue  $E_{n,l}^{(d)}(\lambda)$  of the Hamiltonian

$$H^{(d)} = \frac{1}{2m} \mathbf{p} \cdot \mathbf{p} + \frac{1}{2}m\omega^2 \mathbf{r} \cdot \mathbf{r} + \lambda (\mathbf{r} \cdot \mathbf{r})^2$$

$$H^{(d)}|\psi\rangle = E^{(d)}|\psi\rangle$$

(where  $\mathbf{r}$  and  $\mathbf{p} = -i\nabla$  are position and momentum operators in  $d (= 1, 2, 3)$  dimensions) is nothing but the expectation value of  $H^{(d)}$ , calculated with respect to the basis states of a 'renormalised harmonic oscillator'. The details of how to choose the renormalised frequency  $\omega_0(n, l, \lambda)$  so as to make the expectation value quite close to the actual energy will be discussed in the next section. We may also mention here that the work of Banerjee and co-workers (1978) for the one-dimensional oscillator also rests on the use of renormalised basis states, but their aim is not to obtain a simple analytic formula for  $E_n$  but to compute  $E_n$  numerically to high accuracy<sup>†</sup>.

## 2. Energy values of the pure quartic oscillators

The Hamiltonian for a particle moving in a quartic potential

$$V = \lambda (\mathbf{r} \cdot \mathbf{r})^2 \quad (1)$$

(where  $\mathbf{r} = (r_1, r_2, \dots, r_d)$  and  $d = (1, 2 \text{ or } 3)$ ) is given by<sup>‡</sup>

$$H^{(d)} = \frac{1}{2m} \mathbf{p} \cdot \mathbf{p} + \lambda (\mathbf{r} \cdot \mathbf{r})^2. \quad (2)$$

<sup>†</sup> Killingbeck (1981) has developed a perturbation series method for the one-dimensional AHO using a renormalised harmonic oscillator basis states, but his perturbation series is expected to be divergent. The divergent Rayleigh-Schrödinger series is known to be summable by the method of Padé approximants (Loeffel *et al* 1969), and one of the referees has stated that Austin and Killingbeck have found similar Padé summability for Killingbeck's renormalised perturbation series.

<sup>‡</sup> The results for the potential  $V = \frac{1}{2}m\omega^2 \mathbf{r} \cdot \mathbf{r} + \lambda (\mathbf{r} \cdot \mathbf{r})^2$  will be mentioned at the end.



A scaling argument shows readily that the eigenvalues of (2) are  $(\lambda \hbar^4/m^2)^{1/3}$  times those of  $\frac{1}{2}\mathbf{p} \cdot \mathbf{p} + (\mathbf{r} \cdot \mathbf{r})^2$  and we can therefore assume without loss of generality that  $m = \lambda = \hbar = 1$ . We consider now the matrix representation of  $H^{(d)}$  in a harmonic oscillator basis which diagonalises

$$H_0^{(d)} = \frac{1}{2}\mathbf{p} \cdot \mathbf{p} + \frac{1}{2}\omega_0^2 \mathbf{r} \cdot \mathbf{r} \quad (3)$$

where  $\omega_0$  is a renormalised frequency. The non-vanishing matrix elements of  $H^{(d)}$  are easily determined (Bell 1970):

$$\begin{aligned} \langle n, l | H^{(d)} | n, l \rangle &\equiv H_{nn} \\ &= \left[ \frac{1}{2}(n + \frac{1}{2}d) + \frac{1}{4}\omega_0^{-3} [6n(n+d) - 2l(l+d-2) + d(d+2)] \right] \omega_0 \end{aligned} \quad (4a)$$

$$\begin{aligned} \langle n, l | H^{(d)} | n+2, l \rangle &= \langle n+2, l | H^{(d)} | n, l \rangle \equiv H_{n,n+2} \\ &= [(n-l+2)(n+l+d)]^{1/2} \left[ \frac{1}{4} - \omega_0^{-3} (n + \frac{1}{2}d + 1) \right] \omega_0 \end{aligned} \quad (4b)$$

$$\begin{aligned} \langle n, l | H^{(d)} | n+4, l \rangle &= \langle n+4, l | H^{(d)} | n, l \rangle \equiv H_{n,n+4} \\ &= \frac{1}{4}\omega_0^{-2} [(n-l+2)(n-l+4)(n+l+d)(n+l+d+2)]^{1/2} \end{aligned} \quad (4c)$$

where  $n = 0, 1, 2, \dots$  and for given  $n, l$  takes values<sup>†</sup>  $n, n-2, n-4, \dots, 1$  or  $0$ . The frequency  $\omega_0$  is arbitrary and, as yet, undetermined. For a given  $n$  and  $l$ , we wish to choose  $\omega_0$  in such a way that the diagonal element (4a) closely approximates the actual eigenvalue  $E_{n,l}^{(d)}$ . The fairly obvious idea that this may become possible if we make the matrix elements immediately close to  $H_{nn}$  (namely  $H_{n\pm 2,n}$  and  $H_{n,n\pm 2}$ ) as small as possible leads to the condition

$$\omega_0^3 = 4(n + \frac{1}{2}d). \quad (5)$$

We shall see that with slight modifications of this equation (dictated by what is known about the asymptotic—large- $n$ —limit of  $E_n$  and other considerations) the energy eigenvalues with quite high accuracy are obtained from (4a):

$$E_{n,l}^{(d)} \simeq H_{nn} = \left[ \frac{1}{2}(n + \frac{1}{2}d) + \frac{1}{4}\omega_0^{-3} [6n(n+d) - 2l(l+d-2) + d(d+2)] \right] \omega_0. \quad (6)$$

### 2.1 Refinement in the choice of $\omega_0$

(i) *One-dimensional case* ( $d = 1$ ). In the large- $n$  limit, equations (5) and (6) yield, to the two leading orders in  $(n + \frac{1}{2})$ ,

$$E \simeq 1.389(n + \frac{1}{2})^{4/3} \left( 1 + \frac{3}{28(n + \frac{1}{2})^2} \right). \quad (7)$$

Equation (7) has the same form as the two leading terms in the asymptotic WKB expression (Hioe and Montroll 1975, Pasupathy and Singh 1980). In the latter,

$$E^{(\text{WKB})} \simeq C(n + \frac{1}{2})^{4/3} \left( 1 + \frac{\delta}{(n + \frac{1}{2})^2} + \dots \right)$$

where  $C = 3^{4/3} \pi^2 [\Gamma(\frac{1}{4})]^{-8/3} = 1.376\,5074$  and  $\delta = 1/9\pi = 0.0354$ . The values in (7) namely 1.389 and  $3/28$  are not very far from the corresponding WKB ones. It is readily seen that adjustment of these coefficients can be accomplished by modifying the

<sup>†</sup> For  $d = 2$ , the angular momentum is actually  $\pm l$  but the energy is degenerate with respect to the sign. Clearly  $l = 0$  for the one-dimensional oscillator.



defining equation for  $\omega_0$  to

$$\omega_0^3 = \frac{4(n + \frac{1}{2}d)}{a + c/(n + \frac{1}{2}d)^2}. \quad (8)$$

The large- $n$  limit of  $E$  of equation (6) then retains the same form as (7) but with the two leading coefficients modified to

$$\frac{1}{2}(4/a)^{1/3}(1 + \frac{3}{4}a) \quad (9)$$

and

$$c\left(-\frac{1}{3a} + \frac{3}{4+3a}\right) + \frac{3a}{4(4+3a)} \quad (10)$$

respectively.

Note that only  $a$  is involved in the leading coefficient, and that the second coefficient involves  $c$  linearly. By requiring (9) and (10) to have the WKB values we find the values needed for  $a$  and  $c$  to be

$$a = 0.895\,647\,259 \quad (11)$$

$$c = -0.85. \quad (12)$$

The energies now calculated from (6) and (8) are given in table 1, and as can be seen, they are in good agreement with numerical results of Banerjee *et al* (1978) not only in the region of large  $n$  but over the entire range of  $n$  values. It may be mentioned here that in a recent note by us (Mathews *et al* 1981) a more general formula applicable to anharmonic oscillators in one dimension ( $V = \frac{1}{2}m\omega^2 x^2 + \lambda x^4$ ) was presented. It involves, besides the same  $a$  and  $c$  as above, one more parameter  $b$ . The generalisation of that formula to  $d$  dimensions is given in equations (15) and (16) below.

(ii)  $d=3$ . It is well known that the s-wave levels in a spherically symmetric potential are the same as the odd-parity levels of the one-dimensional system with the

**Table 1.** Energy levels of the pure quartic oscillator in one dimension.

$n$	Energy $E_{n,0}^{(1)}$
0	0.675 (68)
2	4.700 (697)
4	10.244 92 (31)
6	16.712 1 (1 9)
8	23.890 1 (0 0)
10	31.659 54 (46)
20	77.236 11 (08)
48	243.467 364 (38)



same potential. More precisely  $E_{n,0}^{(3)} = E_{n+1}^{(1)}$ . Consequently the wkb limit has the same form as in the one-dimensional case and hence also the same values of the parameters  $a$  and  $c$  in  $\omega_0$  (equation (8)). One sees from the expression (6) that  $E_{n,l}^{(3)}$  decreases as  $l$  increases (for fixed  $n$ ), through a term proportional to  $l(l+1)$ . While this is found to take care of the  $l$  dependence very well for large  $n$ , small deviations from the available numerical results appear when  $n$  is small. An empirical modification, consisting of a change in the defining equation of  $\omega_0$  to

$$\omega_0^3 = \frac{4(n + \frac{1}{2}d)}{a + [c + el(l + d - 2)]/(n + \frac{1}{2}d)^2} \quad (13)$$

with  $e = -0.1$  almost wipes out even these discrepancies, reducing the residual error to less than 1% at the worst (which occurs for small  $n$ ). The form (13) is chosen in order

**Table 2.** Energy levels  $E_{n,l}^{(2)}$  of the pure quartic oscillator in two dimensions.

$n \backslash l$	0	2	4	6	8
0	1.490 (77)				
2	6.001 6 (3 4)	5.671 (24)			
4	11.801 4 (2 4)	11.545 (35)	10.82 (76)		
6	18.458 05 (82)	18.249 3 (5 4)	17.640 (16)	16.68 (60)	
8	25.791 17 (79)	25.613 2 (1 5)	25.088 (76)	24.240 (01)	23.12 (01)
10	33.693 8 (4 3)	33.537 4 (6 6)	33.073 (67)	32.316 (295)	31.292 (33)
20	79.763 7 (4 1)	79.661 5 (1 7)	79.355 84 (19)	78.849 3 (6 3)	78.146 (38)
48	246.823 10 (26)	246.764 9 (5 1)	246.590 41 (64)	246.299 89 (300 07)	245.893 797 (12)
	1	3	5	7	9
1	3.424 (398)				
3	8.704 2 (0 4)	8.15 (09)			
5	14.978 6 (7 8)	14.526 (09)	13.671 (01)		
7	21.999 662 (00)	21.618 7 (1 3)	20.880 (50)	19.83 (74)	
9	29.634 71 (88)	29.304 (00)	28.654 (38)	27.71 (67)	26.52 (40)
11	39.798 09 (35)	37.503 6 (1 5)	36.923 (13)	36.071 (42)	34.972 (02)
21	84.841 9 (2 2)	84.643 92 (86)	84.249 5 (8 2)	83.662 (57)	82.885 (74)
49	253.547 47 (65)	253.432 66 (87)	253.203 20 (43)	252.859 45 (57)	252.401 93 (59)



**Table 3.** The energy levels  $E_{n,l}^{(3)}$  of the pure quartic oscillator in three dimensions.

$n \backslash l$	0	2	4	6	8
0	2.41 (39)				
2	7.335 7 (7 0)	6.879 (30)			
4	13.379 69 (34)	13.019 (05)	12.22 (16)		
6	20.221 0 (0 9)	19.922 (16)	19.225 (18)	18.24 (15)	
8	27.706 49 (39)	27.449 1 (5 9)	26.859 (45)	25.963 (19)	24.80 (69)
10	35.740 380 (12)	31.512 9 (1 0)	34.988 4 (0 2)	34.182 (57)	33.12 (06)
20	82.299 370 (45)	82.148 51 (23)	81.797 7 (6 5)	81.249 8 (5 9)	80.51 (50)
48	250.183 369 (51)	250.096 671 (79)	249.894 504 (45)	249.577 179 (38)	249.145 2 (4 8)
	1	3	5	7	9
1	4.51 (48)				
3	10.108 (00)	9.458 (01)			
5	16.602 (00)	16.067 (46)	15.16 (08)		
7	23.797 6 (6 2)	23.341 (32)	22.546 (08)	21.46 (36)	
9	31.578 86 (08)	31.179 0 (3 6)	30.474 (55)	29.490 (36)	28.26 (14)
11	39.869 49 (01)	39.512 (09)	38.877 (65)	37.982 (49)	36.85 (77)
21	87.392 96 (87)	87.149 3 (8 8)	86.712 6 (0 6)	86.086 1 (0 5)	85.275 (62)
49	256.916 238 (20)	256.773 69 (74)	256.517 316 (38)	256.147 50 (37)	255.664 80 (15)

that the asymptotic expression (for large  $n$ ) should have the same form as with old definition of  $\omega_0$ , except for the necessary small changes in the values of the numerical coefficients appearing in the leading two orders. The values of  $E_{n,l}^{(3)}$  calculated from (6) with  $\omega_0$ ,  $a$ ,  $c$  and  $e$  given by (13), (11) and (12) are tabulated (table 3) and found to be in good agreement over the entire range of  $n$  and  $l$  for which accurate numerical values are available from earlier works<sup>†</sup>.

(iii)  $d = 2$ . The common formula equations (6) and (13) applicable to  $d = 1$  and  $d = 3$  was itself found to be very accurate for  $d = 2$  also, with the parameters  $a$ ,  $c$  and  $e$  having the same values as in (11) and (12). The results are given in table 2.

<sup>†</sup> One of the referees has intimated that according to unpublished calculations by Killingbeck based on the renormalised perturbation series, the results of Bell *et al* (1970b) in the case  $d = 3$  are slightly in error (in the last one or two decimal places given) for  $n > 20$ .



### 3. Discussion

The results for different  $n, l$  in the three cases  $d = 1, 2$  and  $3$  are presented in tables 1, 2 and 3. Only the last one or two digits in the tabulated values differ from the corresponding digits (shown in brackets below each number) of the accurate values obtained by earlier workers through very elaborate computations.

A few observations on our results are in order. Firstly, the estimation of the ground-state and the first-excited-state energy values is not as good as for higher levels. The reason is basically that, while for other values of  $n$  ( $n \geq 2$ ) the effects of the off-diagonal elements  $H_{mm'}$  with  $m + m' < 2n$  are largely counterbalanced by those with  $m + m' > 2n$ , for  $n = 0$  and  $n = 1$  there is an imbalance because no non-vanishing  $H_{mm'}$  with  $m + m' < 2n$  exist. The obvious remedy is to add a correction to  $H_{nn}$  to take account of the off-diagonal elements very close to  $H_{00}$  or  $H_{11}$  as the case may be†. Secondly, as we have seen above, for given  $n$  we have adopted a renormalised frequency  $\omega_0(n)$  by equation (13) and the  $n$ th diagonal element  $H_{nn}$  of the Hamiltonian matrix in the basis of *this* renormalised  $\omega_0$  then closely approximates the actual energy  $E_n$ . The merit of our prescription for choosing  $\omega_0(n)$  is that the smallest eigenvalue  $\varepsilon_n$  of the matrix  $\mathcal{H} = H - H_{nn}I$  (where  $I$  is a unit matrix) turns out to be close to zero thereby showing that the off-diagonal elements of  $H$  around  $H_{nn}$  (in the basis of  $\omega_0(n)$ ) contribute only a small correction to  $H_{nn}$ . In table 4 we exhibit for a typical case,  $n = 45$ ,  $l = 9$  the values of  $\varepsilon_n$  when  $\mathcal{H}$  is truncated to be a  $(2M+1) \times (2M+1)$  matrix with elements

$$\mathcal{H}_{ij} = H_{2i-n, 2j-n} - H_{nn}\delta_{ij} \quad (i, j = n-M, \dots, n+M).$$

We observe from table 4 that as  $M$  increases the value of  $\varepsilon_n$  stabilises, and the corrected energy  $H_{nn} + \varepsilon_n$  coincides with the Bell *et al* (1970b) value‡. Thirdly, for both two- and three-dimensional oscillators, the energy will decrease with  $l$  (for fixed  $n$ ). The formula given by Quigg and Rosner for the wkb regime depends only on  $n + l$ , and as such, for fixed  $n$ , predicts an increase of  $E$  which is incorrect (cf the table of  $E_{n,l}$  computed by Bell *et al*).

**Table 4.** Residual correction to the eigenvalue with  $n = 45$ ,  $l = 9$ .  
 $\mathcal{E}_{45,9} = E_{45,9} + \varepsilon_n = 228.822\,394 + \varepsilon_n$ .

$M$	$\varepsilon_n$	$E_{45,9} + \varepsilon_n$
2	-0.160 332	228.662 062
3	-0.822 558	227.999 836
5	-0.066 226	228.756 168
8	-0.000 192	228.822 202
10	-0.000 960	228.821 434
12	-0.000 988	228.821 406
15	-0.000 997	228.821 397
		(228.821 395)

† What is shown against  $n = 0$  in the tables is not  $H_{00}^{(d)}$  but the lower of the eigenvalues of the matrix  $\begin{pmatrix} H_{00}^{(d)} & H_{02}^{(d)} \\ H_{20}^{(d)} & H_{22}^{(d)} \end{pmatrix}$  in a representation with  $\omega_0$  given by (13) with  $a - 1 = c = e = 0$ . For  $n = 1$  also the effect of the nearest off-diagonal elements  $H_{13}^{(d)}, H_{31}^{(d)}$  has been taken into account in a similar fashion.

‡ A powerful algorithm for estimating  $\varepsilon$  numerically to a high accuracy (one part in  $10^{15}$ ) has been developed by one of us (PMM) the details of which will be published separately.



Finally we give below an expression for the eigenvalues  $E_{n,l}^{(d)}$  of the anharmonic oscillator Hamiltonian in  $d \leq 3$  dimensions.

$$H = p^2/2m + \frac{1}{2}m\omega^2 \mathbf{r} \cdot \mathbf{r} + \lambda (\mathbf{r} \cdot \mathbf{r})^2 \quad (14)$$

$$E_{n,l}^{(d)} = \left\{ \frac{1}{2}\nu^{-1}(n + \frac{1}{2}d)(1 + \nu^2) + \frac{1}{8}\rho^2\nu^2[6n(n+d) - 2l(l+d-2) + d(d+2)] \right\} \hbar\omega \quad (15)$$

where  $\rho^2 = 2\lambda \hbar/m^2\omega^3$  and  $\nu = \omega/\omega_0$  is the solution lying between 0 and 1 of the algebraic equation

$$(2n+d)\rho^2\nu^3 = (1-\nu^2)\left(1 + (a-1)(1-\nu^2) + b\nu^2 + \frac{c+el(l+d-2)}{(n+\frac{1}{2}d)^2}\right) \quad (16)$$

where the values of the constants  $a$ ,  $c$  and  $e$  are the same as in equations (11) and (12) and

$$b = -0.125\ 020.$$

This value of  $b$  was found necessary in order that in the wkb limit, the coefficient of the next to the leading term in  $\lambda$  of the energy (namely  $0.268\ 055\ \lambda^{-1/3}(n+\frac{1}{2})^{2/3}$ ) is also reproduced correctly by (15).

The special case of this formula with  $d=1$  was first proposed by Mathews *et al* (1981) who demonstrated its accuracy over the entire domain of  $n$  and  $\lambda$  values. Also as  $\omega \rightarrow 0$  (for any  $d$ ) equations (15) and (16) do go over into the equations (6) and (13) of the pure quartic case which were considered in detail in this paper. In view of the experience with both these special cases ( $d=1$ ,  $\omega$  arbitrary;  $\omega \rightarrow 0$ ,  $d=1, 2, 3$ ) it is to be expected that the formula (15) will yield results of the same kind of accuracy also for  $d=2, 3$  with  $\omega \neq 0$ . We do not present any numerical results for these anharmonic oscillators, however, as no independently computed values are available in the literature for comparison purposes.

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*Note added in proof.* The  $n$  of Quigg and Rosner is the radial quantum number  $n_r$  and is related to our  $n$  as  $n = 2n_r + l$ . The Quigg-Rosner levels with different sets of values of  $n_r$  and  $l$  but with a fixed  $n$  are therefore degenerate in energy. The  $l$ -dependent term in our formula (6) removes this degeneracy. The  $l$ -dependence of the Quigg-Rosner formula is spurious in the same sense as that of the formula for the hydrogen atom would be if it were written in terms of the radial quantum number as  $\text{constant}/(n_r + l + 1)^2$ .

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